MATHEMATICS

TRANSCENDENTAL VALUES OF A CLASS OF FUNCTIONS

BY

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1. *Introduction.* In this paper we show that each function $f(z)$ belonging to a certain class of multi-valued analytic functions assumes only transcendental values for all algebraic numbers z with the exception of a finite set. In particular we prove the following theorems.

Let α be an algebraic irrational and $\beta = r\alpha + s$ where r and s are arbitrary rationals.

Theorem 1. *Let $F(z)$ be a polynomial of degree at least one with algebraic coefficients such that $F(0)=0$. Set for $n=1, 2, 3, \ldots$*

$$(1.1) \qquad F_n \equiv \begin{cases} (n\alpha+\beta)^{-1}\left[(d/dz)^n(1+F(z))^{n\alpha+\beta}\right]_{z=0} & \text{if } n\alpha+\beta \neq 0 \\ 0 & \text{if } n\alpha+\beta = 0. \end{cases}$$

Then the series $\sum_{n=1}^{\infty} F_n(z^n/n!)$ generates an analytic function $f(z)$. For every algebraic number z, distinct from the zeros of $F(z)$, the function $f(z)$ is regular and $f(z)$ together with all its derivatives has only transcendental values.

Theorem 2. *Set $p^{(n)} = p(p-1)\ldots(p-n+1)$; $p^{(0)}=1$. Let $f_\lambda(z)$ denote the analytic function generated by the power series*

$$(1.2) \qquad \sum_{n=\lambda}^{\infty} (\alpha n+\beta)^{(n-\lambda)}(z^n/n!) \qquad (\lambda = 1, 2, 3, \ldots)$$

where

$$(1.3) \qquad \beta \neq -(\lambda-j)\alpha - k; \quad j=1, 2, \ldots, \lambda; \quad k=1, 2, \ldots, j.$$

For every algebraic number z, different from zero, the function $f_\lambda(z)$ is regular and has only transcendental values.

For the case $F(z)=z$ in Theorem 1 we get $F_n = (n\alpha+\beta-1)^{(n-1)}$ and the series

$$(1.4) \qquad \sum_{n=1}^{\infty} (n\alpha+\beta-1)^{(n-1)}(z^n/n!).$$

This is just (1.2) for $\lambda=1$ and β replaced by $\beta-1$. Therefore, the case for $\lambda=1$ excluded in Theorem 2 is covered by Theorem 1.

Differentiating (1.4) gives

$$\sum_{n=0}^{\infty} (n\alpha+(k+1)\alpha+\beta-1)^{(n+k)}(z^n/n!) \qquad (k=0,1,2,\ldots) \tag{1.5}$$

or, equivalently, the series

$$\sum_{n=0}^{\infty} (n\alpha+\beta')^{(n+k)}(z^n/n!) \qquad (k=0,1,2,\ldots) \tag{1.6}$$

where $\beta'=R\alpha+S$ and R and S are arbitrary rationals. The collection of series (1.6) is not more general than (1.5) since r and s are arbitrary rationals also. The functions generated by (1.6) were considered previously by the author and the result was quoted by POPKEN [1]. Theorem 1 contains the previous result and extends it to a much more general class of functions.

The proof of Theorem 1 is presented in Section 2. It is based on a method developed by POPKEN in [1]. The only results used from the theory of transcendental numbers are the Gelfond–Schneider Theorem (cf., e.g., [2], pp. 75f) on the transcendence of a^b, where $\log a \neq 0$ and b is an algebraic irrational, and the Lindemann–Weierstrass Theorem ([2], pp. 23f).

The proof of Theorem 2, given in Section 3, is basically different from the proof of Theorem 1. In the proof of Theorem 2, we work within the circle of convergence of the power series in obtaining the "closed form" representation of the function and extend the result by continuation (cf. Sec. 3, Part A). However, in the proof of Theorem 1 this is obviated (cf. Sec. 2, Part C).

2. Proof of Theorem 1. In the case $\alpha n+\beta=0$, which occurs only for $\beta=-N\alpha$, we have defined $F_N=0$ in the statement of the theorem. However, it is clear that we could have chosen for F_N any algebraic number. In the following we take $F_N=[(d/dz)^N \log(1+F(z))]_{z=0}$.

A. Consider the function of the complex variable w

$$z=w(1+F(w))^{-\alpha}. \tag{2.1}$$

Let $S=\{u : F(u)=0,\ -1\}$. For every $w\notin S$, $z(w)$ is regular and never vanishes. Moreover, for every algebraic number $w\notin S$, it follows from the Gelfond–Schneider Theorem that $z(w)$ is transcendental. Hence the inverse function $w=w(z)$ has only transcendental values for all algebraic z that are regular points for $w(z)$ with the exception of the zeros of $F(z)$.

B. In this part we shall prove that all the singular points of $w(z)$ are either transcendental numbers or zeros of $F(z)$.

From (2.1) we find that $w(z)$ satisfies the differential equation

$$dw/dz=w(1+F(w))/z(1+F(w)-\alpha wF'(w)). \tag{2.2}$$

The second member of (2.2) is a rational expression of z and w. Hence the singularities of $w(z)$ belong either to the "fixed" or "movable" singular points of (2.2). Clearly the only fixed singular point is $z=0$, a zero of $F(z)$. It follows from a classical theorem of Painlevé [3] that the movable singular points of $w(z)$ are algebraic critical points. A consideration of the equation obtained from (2.2) by the substitution $w=W^{-1}$ shows simply that the functional values of these points must be finite. For these singular points

$$1+F(w)-\alpha wF'(w)=0$$

and

$$1+F(w)\neq 0.$$

Thus the numbers w are algebraic and from (2.1) we deduce that the corresponding z are either transcendental numbers or zeros of $F(z)$.

C. Consider now the function

$$f(z)=\begin{cases}\beta^{-1}[(1+F(w))^{\beta}-1], & \beta\neq 0\\ \log(1+F(w)) &, \beta=0\end{cases}\tag{2.3}$$

where $w=w(z)$ is the inverse of (2.1).

We shall show that the function defined by (2.3) is exactly the function generated by the series in the statement of the theorem.

We choose the branch of $w(z)$ which vanishes at $z=0$ and seek the corresponding branch of $f(z)$. From the inversion formula of Lagrange we have

$$g(w)=g(0)+\sum_{n=1}^{\infty}(z^n/n!)\{(d/dx)^{n-1}[g'(x)(1+F(x))^{n\alpha}]\}_{x=0}\tag{2.4}$$

for any function $g(w)$ regular in a neighborhood of $w=0$.

Applying (2.4) to the branch of $g(w)=\beta^{-1}[(1+F(w))^{\beta}-1]$ which vanishes at $w=0$ gives

$$\sum_{n=1}^{\infty}(z^n/n!)(n\alpha+\beta)^{-1}[(d/dx)^n\,(1+F(x))^{n+\beta}]_{x=0}=\sum_{n=1}^{\infty}F_n(z^n/n!)$$

for $\beta\neq -N\alpha$, N a positive integer. For $\beta=-N\alpha$ we get for the coefficient of $z^N/N!$

$$(d/dx)^N\log(1+F(x))|_{x=0}=F_N$$

and the other coefficients the same as above.

For the case $\beta=0$ we apply (2.4) to the branch of $\log(1+F(w))$ which vanishes at $w=0$ and get, as above, $\sum_{n=1}^{\infty}F_n(z^n/n!)$.

In all cases we obtain the branch $\sum_{n=1}^{\infty}F_n(z_n/n!)$. Thus the function $f(z)$ as defined in (2.3) is identical with the function introduced in the statement of the theorem. In Part B we have shown that the singular points of $w(z)$ are either zeros of $F(z)$ or transcendental numbers. Thus it follows that $f(z)$ is regular for every algebraic number z distinct from the zeros of $F(z)$.

D. To complete the proof for the function $f(z)$ we assume that for some algebraic number z_0, $F(z_0) \neq 0$, at least one of the functional values $f(z_0)$ is algebraic. This assumption leads us to a contradiction as we will demonstrate.

From (2.3) it follows that either $\beta^{-1}[(1+F(w)^{\beta}-1]$ or $\log(1+F(w))$ is algebraic.

In the first case set $\beta^{-1}[(1+F(w))^{\beta}-1]=a$ (algebraic). Since $\beta=r\alpha+s$, it follows that

$$(1+F(w))^{r\alpha}(1+F(w))^{s}=\beta a+1$$

and from (2.1)

$$(z_0/w)^{r}(1+F(w))^{s}=\beta a+1$$

where r and s are rationals. Hence $w(z_0)$ has at least one algebraic functional value. This contradicts the results of parts A and B.

In the second case we deduce from $\log(1+F(w))=a$ and (2.1) that $w=z_0e^{a\alpha}$. This combined with $1+F(w)=e^{a}$ gives the relation

$$1+F(z_0e^{a\alpha})=e^{a}.$$

However, this is in contradiction to the assertion of the Lindemann–Weierstrass Theorem which states that if $c_1, c_2, \ldots, c_k$ are different algebraic numbers, then $\sum_{j=1}^{k} b_je^{c_j} \neq 0$ for every set of algebraic numbers $b_1, b_2, \ldots, b_k$, not all zero.

Thus the theorem is established for $f(z)$.

E. We shall now prove the assertion concerning the derivatives of $f(z)$. Set

$$G(w)=1+F(w)-\alpha wF'(w). \tag{2.5}$$

If $F(w)=a_kw^k+\ldots+a_1w$, then the coefficient of w^k in $G(w)$ is $a_k(1-\alpha k)$. It is not zero since α is irrational. Thus the degrees of $G(w)$ and $F(w)$ are the same, namely k.

Now from (2.1), (2.2) and (2.5) $dw/dz=(1+F(w))^{\alpha+1}(G(w))^{-1}$. Then it follows from (2.3) that

$$f'(z)=F'(w)(1+F(w))^{\alpha+\beta}(G(w))^{-1}. \tag{2.6}$$

By repeated differentiation of (2.6) we obtain the relation

$$f^{(n)}(z)=(1+F(w))^{n\alpha+\beta}(G(w))^{-2n+1}P_n(w) \tag{2.7}$$

where

$$P_1(w)=F'(w)$$

and

$$\begin{aligned} P_n(w)=[(n-1)\alpha+\beta]&G(w)P_{n-1}(w)F'(w)\\ &-(2n-3)(1+F(w))G'(w)P_{n-1}(w)\\ &+(1+F(w))G(w)P'_{n-1}(w). \end{aligned}$$

The polynomial $P_n(w)$ is of degree $(2n-1)k-n$ at most. Moreover, $P_n(w)$ cannot be identically zero. For if this was the case, then $P_{n+m}(w)$, $m \geqq 1$, would be identically zero also. Then it would follow from (2.7) that $f(z)$ is a polynomial. However, this is in contradiction to the results of parts A through D.

Now let z_0 be an algebraic number such that $F(z_0) \neq 0$. Assume that at least one of the functional values $f^{(n)}(z_0)$ is algebraic. Let w_0 be the associated value of $w(z_0)$. Thus the expression from (2.7)

$$(1+F(w_0))^{(n+r)\alpha+s}(G(w_0))^{-2n+1}P_n(w_0)$$

is algebraic, where $r\alpha+s=\beta$. From (2.1)

$$(w_0/z_0)^{n+r}=(1+F(w_0))^{(n+r)\alpha}.$$

Since $z_0{}^{n+r}$ is algebraic, it follows from these expressions that

$$(1+F(w_0))^s w_0{}^{n+r}P_n(w_0)(G(w_0))^{-2n+1}$$

is an algebraic number, say a.

Consider the equation

$$(1+F(x))^s x^{n+r}P_n(x)(G(x))^{-2n+1}=a. \tag{2.8}$$

We must show that (2.8) is not an identity, i.e., that the left member is not identically a constant. It is clear that $G(x)$ has at least one zero, say $\bar{x}$, distinct from the zeros of $1+F(x)$. Since $G(0)=1$, $\bar{x}\neq 0$. Let $\bar{x}$ be of order d. It is not difficult to show that $\bar{x}$ is then a zero of $P_n(x)$ with order at most $(n-1)(d-1)$. Since $(2n-1)d>(n-1)(d-1)$, we conclude that the left member of (2.8) is not identically a constant.

Therefore, since w_0 is a root of (2.8), it follows that w_0 must be algebraic. However, we know from parts A through C that w_0 is transcendental.

This completes the proof of Theorem 1.

3. Proof of Theorem 2. We consider here the function of the complex variable w

$$z=w(1+w)^{-\alpha}. \tag{3.1}$$

It follows from Section 2, parts A and B, with $F(w)=w$, that the inverse function $w=w(z)$ has only transcendental values for all algebraic $z\neq 0$ and that $w(z)$ has no finite singularities for $z\neq 0$, $\alpha^{-1}(1-\alpha^{-1})^{\alpha-1}$.

A. Here we shall establish for any $z\neq 0$, $\alpha^{-1}(1-\alpha^{-1})^{\alpha-1}$ the representation

$$f_\lambda(z)=P_\lambda(w)(1+w)^{\beta+1}+Q_\lambda(z) \qquad (\lambda=1,2,\ldots) \tag{3.2}$$

where $w=w(z)$ is a suitably chosen functional value of the function $w(z)$ defined by (3.1) and P_λ and Q_λ are polynomials of degree $\lambda-1$ at most with algebraic coefficients depending on α and β.

We prove (3.2) for the branch of $f_\lambda(z)$ given by the series (1.2) and then the relation follows, in general, by analytic continuation. The proof proceeds by induction on λ.

For the branch of $w=w(z)$, $w(0)=0$, we obtain from (2.4) the representation

$$(3.3) \qquad (1+w)^{\beta+1}=1+(\beta+1)\sum_{n=1}^{\infty}(\alpha n+\beta)^{(n-1)}(z^n/n!)$$

for the branch of $(1+w)^{\beta+1}$ which equals one at $w=0$. From (3.3) we see that relation (3.2) holds for $\lambda=1$.

To complete the proof consider the indefinite integral

$$\int p(x)\,(1+x)^r dx, \quad r\neq -1,\ -2,\ \ldots,\ -\lambda-1$$

where $p(x)$ is a polynomial of degree λ at most. By successive integration by parts

$$(3.4) \qquad \int p(x)\,(1+x)^r dx = P(x)\,(1+x)^{r+1}$$

where

$$(3.4') \quad P(x)=\sum_{k=0}^{\lambda}(-1)^k(r+1)^{-1}\ldots(r+k+1)^{-1}p^{(k)}(x)(1+x)^k,\ p^{(0)}(x)\equiv p(x)$$

is of degree λ at most. Moreover,

$$(3.5) \qquad P(-1)=(r+1)^{-1}\,p(-1).$$

Now suppose that (3.2) holds for some value λ. For the branch of $f_\lambda(z)$ given by the series (1.2) we write (3.2) as

$$(3.6) \quad \begin{cases} \displaystyle\sum_{n=\lambda}^{\infty}(\alpha n+\alpha+\beta)^{(n-\lambda)}(z^n/n!) = \overline{P}_\lambda(w)\,(1+w)^{\beta+\alpha+1}+\overline{Q}_\lambda(z), \\ \qquad \beta\neq -(\lambda+1-j)\alpha-k;\ j=1,2,\ldots,\lambda+1;\ k=1,2,\ldots,j \end{cases}$$

where $\overline{P}_\lambda$ and $\overline{Q}_\lambda$ denote the functions obtained from P_λ and Q_λ, respectively, be replacing β by $\alpha+\beta$.

Integrating both sides of (3.6) and using (3.1) gives

$$\sum_{n=\lambda}^{\infty}(\alpha n+\alpha+\beta)^{(n-\lambda)}(z^{n+1}/(n+1)!) = \int \overline{P}_\lambda(w)(1+w-\alpha w)\,(1+w)^\beta dw + \int \overline{Q}_\lambda(z)dz+\text{constant};$$

and from (3.4) and (3.5)

$$(3.7)\ \sum_{n=\lambda}^{\infty}(\alpha n+\alpha+\beta)^{(n-\lambda)}(z^{n+1}/(n+1)!)=P(w)\,(1+w)^{\beta+1}+\int \overline{Q}_\lambda(z)dz+\text{constant}$$

where $P(w)$ is a polynomial of degree λ at most with algebraic coefficients and $P(-1)=\alpha(\beta+1)^{-1}\overline{P}_\lambda(-1)$. The constant is determined by the condition that $w=0$ for $z=0$. Hence the constant is equal to $-P(0)$, an algebraic number.

It follows from (3.7), setting $P_{\lambda+1}(w) \equiv P(w)$ and $Q_{\lambda+1}(z) \equiv \int \bar{Q}_\lambda(z)dz - P(0)$, that (3.2) holds for $\lambda+1$ and the branch of $f_\lambda(z)$ given by (1.2).

B. To complete the proof of the theorem let $z_0 \neq 0$ be an algebraic number; hence $z_0 \neq \alpha^{-1}(1-\alpha^{-1})^{\alpha-1}$, since it follows from the Gelfond–Schneider Theorem that for algebraic-non-rational α the numbers $\alpha^{-1}(1-\alpha)^{\alpha-1}$ are transcendental. Assume that one of the functional values $f_\lambda(z_0)$ is algebraic.

From (3.2) it follows that $P_\lambda(w_0)(1+w_0)^{r\alpha+s+1}$ is algebraic where w_0 is the suitably chosen value of $w(z_0)$. Since from (3.1) $(1+w_0)^\alpha = w_0/z_0$ and r is rational, the number a defined by

$$P_\lambda(w_0)w_0{}^r(1+w_0)^{s+1} = a \tag{3.8}$$

must be algebraic.

Now the function $P_\lambda(u)u^r(1+u)^{s+1}$ is not identically a. For if r and s are non-positive integers, it follows from (1.3) that $r+s \leqq -\lambda-1$ or $r+s+1 \leqq -\lambda$. However, $P_\lambda(u)$ is a polynomial of degree $\lambda-1$ at most.

Thus relation (3.8) implies that w_0 is algebraic. This is in contradiction to the conclusions in Section 2, parts A and B, applied with the function $F(w) = w$.

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